Grade 7/8 Math Circles
October 18/19/20/24, 2022
Radians

Degrees: Why 360?

Degrees are a unit of measuring angles, denoted by the symbol °.

Stop and Think
How many degrees are there in a full circle? Why?

It is unclear when and by whom exactly 360 degrees was chosen, but there are many theories as to why it was chosen.

Theory 1: Highly Composite

An integer is **divisible** by another integer if that division has no remainder, that is, the number that results from the division is a whole number. For example, 6 is divisible by 2 because $6 \div 2 = 3$ and 3 is a whole number.

A **divisor** is a number that will divide evenly into another number, that is, has no remainder. For example, 2 is a divisor of 6 because $6 \div 2 = 3$ and 3 is a whole number.

There are 24 positive divisors of 360: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360.

Exercise 1
Choose a whole number from 1 to 500 that you think would be a better choice for the circle to be split into. Find the number of positive divisors and compare that number with the 24 divisors of 360.

A number is **highly composite** if it is a positive whole number with more divisors than any smaller positive whole number. The first fifteen highly composite numbers are 1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, and 840.

1	1	①	13	1, 13	2
2	1, 2	②	14	1, 2, 7, 14	4
3	1, 3	2	15	1, 3, 5, 15	4
4	1, 2, 4	③	16	1, 2, 4, 8, 16	5
5	1, 5	2	17	1, 17	2
6	1, 2, 3, 6	④	18	1, 2, 3, 6, 9, 18	6
7	1, 7	2	19	1, 19	2
8	1, 2, 4, 8	4	20	1, 2, 4, 5, 10, 20	6
9	1, 3, 9	3	21	1, 3, 7, 21	4
10	1, 2, 5, 10	4	22	1, 2, 11, 22	4
11	1, 11	2	23	1, 23	2
12	1, 2, 3, 4, 6, 12	⑥	**24**	1, 2, 3, 4, 6, 8, 12, 24	⑧

So, highly composite numbers have a lot of divisors, relative to the size of the number. This means that, when compared to other numbers, more divisions of these numbers result in whole numbers instead of fractions.

Think back to a time long, long ago before calculators and computers, when all math had to be done mentally or on paper. Computations are much easier when working with whole numbers instead of fractions. With 360, a half is 180, a quarter is 90, and a third is 120, all nicer numbers. However, with 100, a half is 50, a quarter is 25, but a third is $\frac{100}{3} = 33.3...$ since 100 is not divisible by 3.

Figure 1: Retrieved from Beautiful Science.

And 360 isn't just divisible by 2, 3, and 4, in fact, 360 is divisible by every whole number from 1 to 10 except 7. So, we have nice numbers for a fifth of a circle, a sixth of a circle, an eighth of a circle, and so on.

Theory 2: Solar and Lunar Years

The standard calendar used at present is the Gregorian calendar. The Gregorian calendar is a **solar calendar**, which means that it is based on the time it takes the Earth to revolve once around the Sun. A solar year is approximately 365 days, or, more precisely, 365.242 days.

In contrast, the **lunar calendar** is based on the phases of the Moon. It takes approximately 29.5 days for the moon to cycle through its phases and a lunar year is 12 full cycles of lunar phases. So, there are about 354 days in a lunar year.

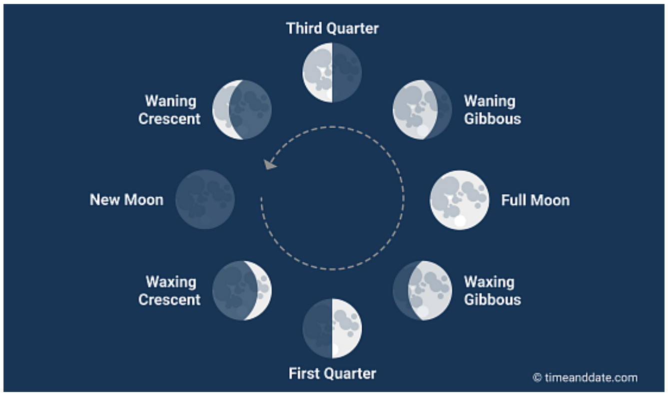


Figure 2: Retrieved from timeanddate.

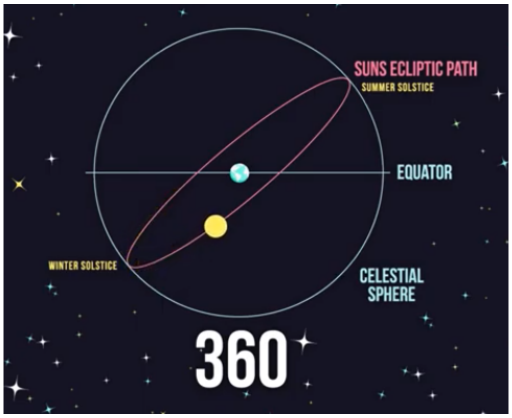


Figure 3: Retrieved from Beautiful Science.

Stop and Think

Which highly composite number falls almost halfway between 354 and 365?

At the time when degrees were created, it was believed that the Sun revolved around the Earth. People at the time noticed that it took about 360 days for the Sun to return to the same position in the sky. Since they believed the Sun revolved around the Earth, it seemed as though the sun moved around in a circle, moving one degree per day. So, it must have seemed reasonable to divide a circle up similar to the way a year is divided up.

Theory 3: Sexagesimal Number System and Equilateral Triangles

The Sumerians and the Babylonians used a **sexagesimal** number system, meaning that their number system used 60 symbols as opposed to our 10 digit number system. The number 60 was a good choice because, like 360, it is highly composite.

Taking an equilateral triangle, whose side length is the distance from the centre to the edge of a circle (radius), and putting one point of the triangle at the centre of the circle and the other points on the edge of the circle (circumference), six such triangles fit in the circle.

Each of these triangles was given a value of 60, since the Babylonians used a sexagesimal system, so all together $6 \times 60 = 360$.

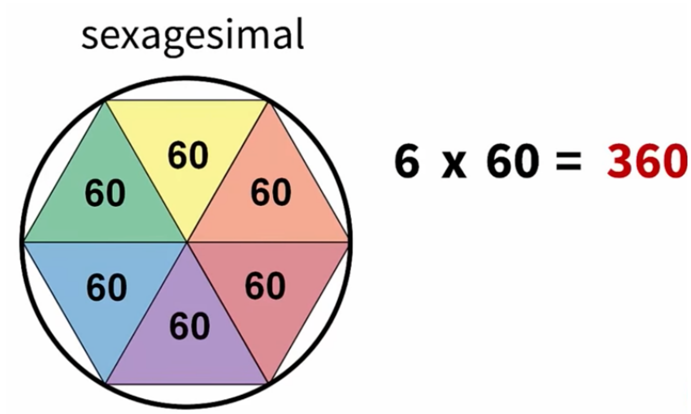


Figure 4: Retrieved from NinetyEast.

Radians

> **Stop and Think**
> Now that we've gone through some possible reasons for why there are 360° in a circle, the next question to ask is why degrees?

Just like measuring height in centimetres or feet and distance in kilometres or miles, angles can be measured using different units.

Before we can explore a new unit for measuring angles, we must first learn a little about circles.

Circles and Pi

The **circumference** is the length of the outer edge of the circle.

An **arc** is any section of the outer edge of the circle.

The **diameter** is the length of any line segment from one point on the circumference to another that passes through the centre of the circle.

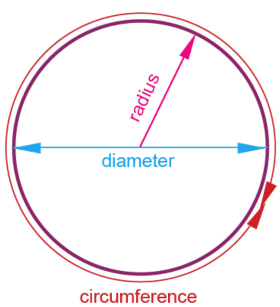
The **radius** (plural: radii) is the length of any line segment from a point on the circumference to the centre of the circle. The radius is half of the diameter.

A **sector** is a part of a circle between two radii and an arc.

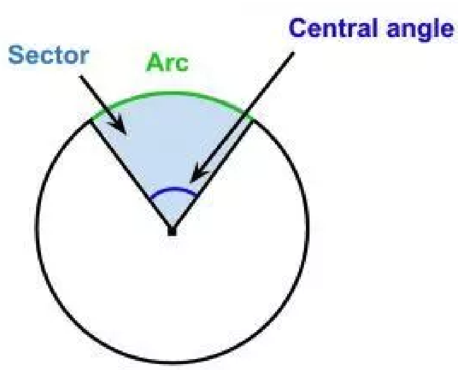
A **minor arc** or **minor sector** is one that covers less than half of the circumference or area, respectively.

A **major arc** or **major sector** is one that covers more than half of the circumference or area, respectively.

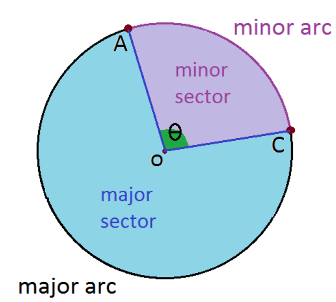
A **central angle** is an angle within a sector whose vertex is at the centre of the circle.



(a) Retrieved from TheSchoolRun.



(b) Retrieved from APEX.



(c) Retrieved from Mathspace.

Figure 5: Labeled Circles

Pi (pronounced “pie” and represented by the Greek letter π) is a very special mathematical number. Try pressing the π button on your calculator, you should get approximately 3.141592654. Most people know that pi is about 3.14 and calculators know a few more decimal places, but pi is actually a non-repeating and non-terminating decimal number, meaning that after the decimal point, the digits continue on forever without any repeating pattern.

Pi is the ratio of a circle’s circumference to its diameter. So, if you divide the circumference by the diameter of any circle, you should get approximately pi. Regardless of the size of the circle, the ratio will always be pi.

If C is the circumference and r is the radius, then the formula for the circumference of a circle is $\boldsymbol{C = 2\pi r}$. Also, since the radius is half of the diameter, if d is the diameter then $2r = d$ and so $\boldsymbol{C = \pi d}$.

If A is the area and r is the radius, then the formula for the area of a circle is $\boldsymbol{A = \pi r^2}$.

Note that:

- A number directly in front of a variable represents multiplication between the number and the variable. For example, $2r = 2 \times r$.
- An exponent (a number up and to the right of another number) represents the repeated multiplication of a single number. For example, $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.
- When simplifying an expression we always compute by the brackets, then exponents, then division and multiplication, then addition and subtraction (BEDMAS).

Definition of Radians

At the start of this lesson, we considered various reasons as to why 360 was chosen as the number to split a circle into. Whatever the reasons, 360 is still just a random number that was chosen by humans.

What if there was a way to define a unit for measuring angles that doesn’t involve just choosing a random number and working from there but rather defining the unit of measurement based on the properties of circles?

Exercise 2

Draw a circle and a radius. Take a string and cut it to a length which is equal to the radius. Lay the string along the circumference of the circle and draw two radii from the centre of the circle to the points on the edge where the ends of the string are. Measure the central angle of the resulting minor sector.

A **radian** is a unit for measuring angles where one radian is the central angle of a sector whose arc is equal to the radius.

A radian is a fixed angle even though it is based on the radius of a circle which can be many different sizes. One radian is always equal to approximately 57.2958°.

Note that there are a lot of symbols and notation used for radians, but in this lesson we will either say radians or use the notation 'rad'.

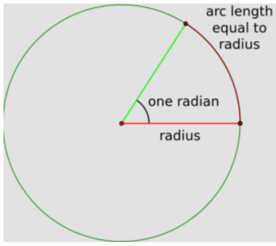


Figure 6: Retrieved from toppr.

An angle (θ) measured in radians is equal to the length of the arc (a) divided by the radius (r).

$$\boldsymbol{\theta = \frac{a}{r}}$$

So, if the length of the arc is equal to the radius, then the angle is equal to $\frac{r}{r} = 1$ rad, just like in our above definition of radians.

If the length of the arc is equal to the circumference, $C = 2\pi r$, then $\theta = \frac{2\pi r}{r} = 2\pi$. So, 2π rad $= 360°$ and there are 2π radians in a full circle.

If the length of the arc is equal to half of the circumference, $\frac{1}{2} \times (2\pi r)$, then $\theta = \frac{\pi r}{r} = \pi$. So, π rad $= 180°$ and there are π radians in a half circle.

Degrees vs Radians

Degrees are useful in our everyday lives and when learning the basics of geometry and angles. However, radians are more useful than degrees when it comes to mathematics.

As mentioned earlier, radians are based on the different parts of a circle, rather than choosing a random number by which to divide up a full circle. As a result, radians make a lot of formulas in mathematics simpler. There are many examples of this in areas of mathematics such as trigonometry and calculus.

In this lesson we will focus on how radians simplify the formulas for the arc length and sector area of a circle.

Conversions

Degrees and radians are both used to measure angles. We already know that 1 radian is equal to approximately 57.296° and that 2π radians are equal to 360° (a full circle).

Stop and Think

How many radians are in 60°? How many degrees are in $\frac{\pi}{6}$ radians?

Multiplying a number by *one* does not change its value. We can use this idea in conversions to make sure that the values stay the same while the units change. In our conversions, we will write the units in our equations because this will help with keeping track of the units.

First, we will convert from degrees to radians. Any number divided by itself is equal to one. Since $\pi \text{ rad} = 180^\circ$, we know that $\frac{\pi \text{ rad}}{180^\circ} = 1$. So, multiplying an angle in degrees by $\frac{\pi \text{ rad}}{180^\circ}$ is the same as multiplying by one and won't affect the value, however the degrees will cancel and we are left with an angle in radians.

Example 1

Convert 60° to radians.

To convert from degrees to radians, we multiply by $\frac{\pi \text{ rad}}{180^\circ}$: $\quad 60^\circ \times \dfrac{\pi \text{ rad}}{180^\circ} = \dfrac{\pi}{3} \text{ rad}$

So, 60° is equal to $\frac{\pi}{3}$ rad.

Exercise 3

Convert the following to radians.

a) $3°$ b) $45°$ c) $90°$ d) $341°$ e) $360°$ f) $450°$

Converting from radians to degrees is very much the same. Since π rad $= 180°$, we know that $\frac{180°}{\pi \text{ rad}} = 1$. So, multiplying an angle in radians by $\frac{180°}{\pi \text{ rad}}$ is the same as multiplying by one and won't affect the value, however the radians will cancel and we are left with an angle in degrees.

Example 2

Convert $\frac{\pi}{6}$ rad to degrees.

To convert from radians to degrees, we multiply by $\frac{180°}{\pi \text{ rad}}$:

$$\begin{aligned}\frac{\pi}{6} \text{ rad} \times \frac{180°}{\pi \text{ rad}} &= \left(\frac{180}{6}\right)^{\circ} \\ &= 30°\end{aligned}$$

So, $\frac{\pi}{6}$ rad is equal to $30°$.

Exercise 4

Convert the following to degrees.

a) $\frac{\pi}{30}$ rad b) $\frac{\pi}{7}$ rad c) π rad d) $\frac{3\pi}{2}$ rad e) 3π rad f) $\frac{7\pi}{9}$ rad

In summary,

- To convert an angle measured in degrees into radians, we multiply by $\frac{\boldsymbol{\pi} \textbf{ rad}}{\textbf{180°}}$.
- To convert an angle measured in radians into degrees, we multiply by $\frac{\textbf{180°}}{\boldsymbol{\pi} \textbf{ rad}}$.

In both cases, we are multiplying by one since $180° = \pi$ rad. Also, we cancel out the units in which the angle is measured by having those units in the denominator of the conversion fraction.

Arc Length

The **arc length** is the distance along the arc. Recall that the arc is any section of the outer edge of the circle.

To see why radians are more useful when calculating arc length, we will first go through an example in degrees.

Example 3 - Degrees

Consider a circle whose radius is 11 cm and a sector of this circle whose central angle is 80°. Calculate the arc length of the sector.

First, we find the circumference: $C = 2\pi r = 2\pi(11) = 22\pi$. So, the circumference is 22π cm. The circumference can be considered the arc length of the 'sector' with central angle 360°. So, since we want the length of the arc of the sector with central angle 80°, if we consider the circumference as 360 sections, then we want 80 sections out of the 360. To find this, we multiply the circumference by $\frac{80}{360}$. So, the arc length is $22\pi \times \frac{80}{360} = \frac{44\pi}{9}$ cm.

Earlier, we defined the following formula where θ is the central angle **measured in radians** of the sector, a is the arc length, and r is the radius: $\theta = \frac{a}{r}$

We can rearrange the above formula to get the arc length formula.

$$\boldsymbol{a = r \times \theta}$$

Example 3 - Radians

Calculate the arc length of a sector with radius 11 cm and a central angle of $80° \times \frac{\pi \text{ rad}}{180°} = \frac{4\pi}{9}$ rad.

$$\begin{aligned} \text{arc length} &= 11 \times \frac{4\pi}{9} \\ &= \frac{44\pi}{9} \end{aligned}$$

So, the arc length is $\frac{44\pi}{9}$ cm. This calculation was simpler than using degrees.

Exercise 5

Calculate the arc length of sectors with the following central angles and radii.

a) radius = 52 cm, central angle = 2π rad

b) radius = 97 mm, central angle = 1 rad

c) radius = 4 m, central angle = $\frac{17\pi}{8}$ rad

d) radius = 8 mm, central angle = $\frac{9\pi}{25}$ rad

Sector Area

We will now look at the **sector area**, which is just the area of a sector.

To see why radians are more useful when calculating sector area, we will first go through an example in degrees.

Example 4 - Degrees

Consider a circle whose radius is 11 cm and a sector of this circle whose central angle is 80°. Calculate the area of the sector.

First, we find the area of the whole circle: $A = \pi r^2 = \pi(11)^2 = 121\pi$. So, the area is 121π cm^2. The area of the whole circle can be considered the area of the 'sector' with central angle 360°. So, since we want the area of the sector with central angle 80°, if we consider the area of the circle as 360 slices, then we want 80 slices out of the 360. To find this, we multiply the area of the whole circle by $\frac{80}{360}$. So, the sector area is $121\pi \times \frac{80}{360} = \frac{242\pi}{9}$ cm^2.

For the area of a sector, we have the following formula where θ is the central angle **measured in radians** of the sector, r is the radius, and A_S is the area of the sector.

$$\boldsymbol{A_S = \frac{1}{2}r^2\theta}$$

Example 4 - Radians

Calculate the area of a sector with radius 11 cm and a central angle of $80^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{4\pi}{9}$ rad.

$$\begin{aligned} A_S &= \frac{1}{2} \times 11^2 \times \frac{4\pi}{9} \\ &= \frac{121 \times 4\pi}{18} \\ &= \frac{242\pi}{9} \end{aligned}$$

So, the sector area is $\frac{242\pi}{9}$ cm^2. This calculation was simpler than using degrees.

Exercise 6

Calculate the area of sectors with the following central angles and radii.

a) radius = 12 cm, central angle = 2π rad

b) radius = 6 mm, central angle = 4 rad

c) radius = 2 m, central angle = $\frac{5\pi}{3}$ rad

d) radius = 9 mm, central angle = $\frac{4\pi}{5}$ rad

Gradians

As we saw earlier, radians are based on the properties of circles and degrees resulted from choosing a somewhat random number.

Stop and Think

Was 360 the only number that has been chosen to split a circle into and make units throughout history? What other numbers do you think could have been chosen?

There is a French system in which 400 was chosen. The units of this system are called **gradians**.

So, 400 gradians is equal to 360 degrees and is also equal to 2π radians.

Exercise 7

What are the following angles in degrees and in radians?

a) 200 gradians

b) 100 gradians

Scientific Calculators

Now, we've learned about three units for measuring angles and we'll likely have to do calculations using these different units. A lot of these calculations are nicer to do using a calculator, but how does the calculator know what units you are using for angles? Scientific calculators have settings that tell them the units.

Changing these settings can be different depending on the calculator. But for this calculator (one allowed in UW), you press the DRG button and move the underline to whichever units you will be using (DEG, RAD, GRD) and hit enter (the equal sign). Then, you will see the unit on the lower part of the screen of your calculator.

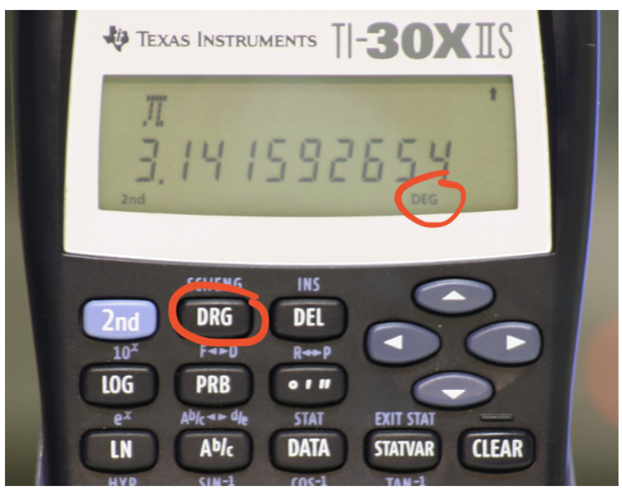


Figure 7: Retrieved from Wikimedia Commons.

Most errors when working with angles are from using a calculator that isn't set for the units which are being used. It is very important to make sure your calculator is on the right setting because otherwise you will end up with different values.